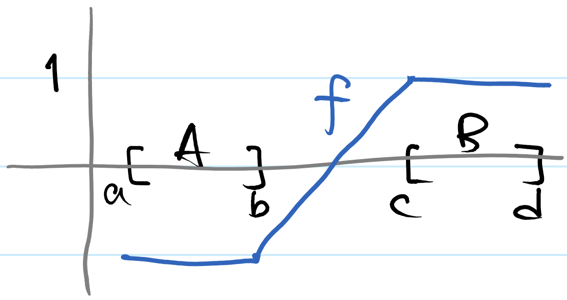
L09 Jan 22 Tietz

Wednesday, January 21, 2015 10:29 AM

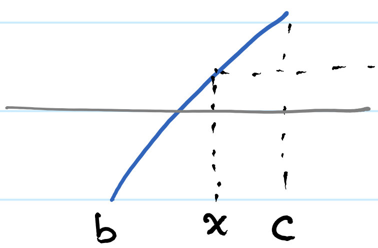
Proposition On a metric space (X, d), if $A, B \subset X$ are disjoint closed sets then $\exists$ continuous $f : X \longrightarrow [0,1]$ such that

$$f|_A \equiv -1 \quad \text{and} \quad f|_B \equiv 1$$

Let us think about $X = \mathbb{R}$, A, B are intervals



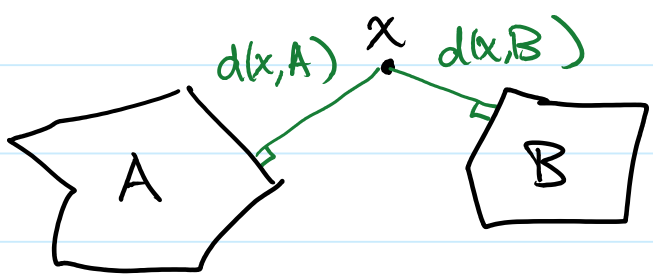
Qu. How would you find the formula for f?



$$-1 + \frac{2(x-b)}{c-b}$$

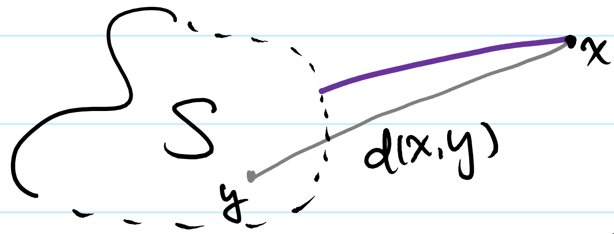
$$= \frac{(x-b)-(c-x)}{c-b}$$

Qu. Does this give us a hint for metric space?



$$f(x) = \frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}$$

Qu. What exactly is $d(x,S)$?



$$d(x,S) = \inf\{d(x,y) : y \in S\}$$

① $d(x,S) = 0 \Rightarrow x \in \overline{S}$

$\inf\{d(x,y) : y \in S\} = 0$, i.e., $\forall\ \varepsilon > 0$

$\exists\ y \in S$ $d(x,y) < \varepsilon$ — $y \in B(x,\varepsilon)$

$\forall$ nbhd U of x, $\underbrace{\exists\ y \in S,\ y \in U}_{S \cap U \neq \phi}$

Since both A, B are closed and $A \cap B = \phi$,

$d(x,A) + d(x,B) \neq 0$

$\therefore$ f is well-defined

② For fixed $y \in S$, $x \longmapsto d(x,y)$ is continuous

Exercise Need the Δ-inequality

Then $\inf\{d(x,y) : y \in S\}$ is also continuous

Tietz Extension

Let X be a metric space and $F \subset X$ be closed; $f : F \longrightarrow [-a,a]$ be continuous.

Then $\exists$ continuous extension $\tilde{f} : X \longrightarrow [-a,a]$

i.e., $\tilde{f}|_F = f$

Idea of proof

Write $[-a,a] = \underbrace{[-a,\frac{-a}{3}] \cup [\frac{-a}{3},\frac{a}{3}] \cup [\frac{a}{3},a]}_{\text{each has length } 2/3}$

$A = f^{-1}[-a,\frac{-a}{3}] \subset F$

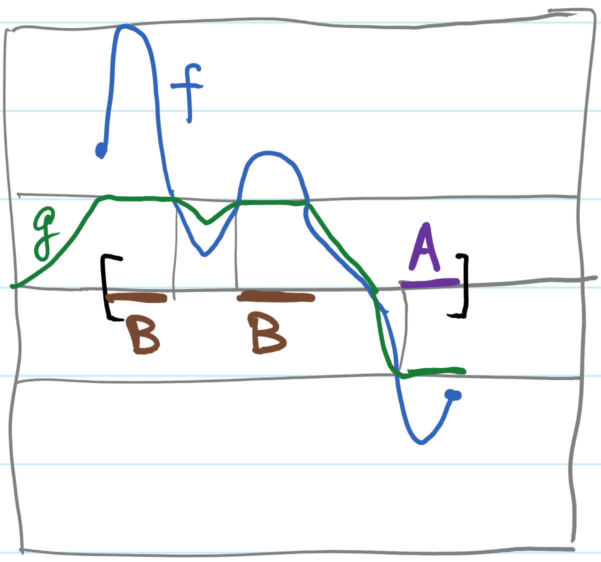
$B = f^{-1}[\frac{a}{3},a] \subset F$

Both are closed in X

and $A \cap B = \emptyset$

$\exists\ g : X \longrightarrow [\frac{-a}{3},\frac{a}{3}]$

$g|_A = \frac{-a}{3}$, $g|_B = \frac{a}{3}$



For future convenience, call it g_1

$|g_1| \leq \frac{a}{3}$ on X

$|f - g_1| \leq \frac{2a}{3}$ on F

Now, we have a continuous

$$(f-g_1) : F \longrightarrow \left[\frac{-2a}{3}, \frac{2a}{3}\right]$$

Repeat the argument to have continuous

$$g_2 : X \longrightarrow \left[\frac{-a}{9}, \frac{a}{9}\right]$$ and

$$\|(f-g_1)-g_2\| \leq \left(\frac{4}{9}\right)a \quad \text{on } F$$

Inductively, we have continuous

① $$g_n : X \longrightarrow \left[\frac{-a}{3^n}, \frac{a}{3^n}\right]$$

② $$\left\|f - \sum_{k=1}^{n} g_k\right\| \leq \left(\frac{2}{3}\right)^n a \quad \text{on } F$$

Note that $\frac{a}{3^n}$, $\left(\frac{2}{3}\right)^n a$ are indep. of $x \in X$ or F

By ①, $\sum_{k=1}^{n} g_k$ converges uniformly to $\tilde{f}$ on X

By ② $f \equiv \tilde{f}$ on F

Remark

Crucial property

$$X \xrightarrow{\text{continuous}} [-1, 1]$$
$$A \longmapsto -1$$
$$B \longmapsto 1$$

Here

Metric

Urysohn Lemma

Normal space